# Engineering Notes

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## Elastostatic Problems of Arbitrarily Shaped Doubly Connected Plates

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### Introduction

STRUCTURAL plates having holes and being irregularly shaped are unavoidable. Especially so when they are used in structures where weight is a premium such as airplanes or space ships. No general solution dealing with elastostatic problems that applies to a variety of complicated geometries and with mixed boundary conditions is known to the writers. In this Note, a general solution of the problem of an arbitrarily shaped, doubly connected plate subjected to a state of plane stress at the boundaries is given. The method used which is called "least square," appears to be far superior than other established numerical methods in treating irregular shapes and with mixed boundary conditions. Some numerical examples are given and compared with published ones where available.

### Formulation

Solution of elastostatic problems involves finding the general solution of the biharmonic differential equation

$$\nabla^2\nabla^2\phi = 0 \tag{1}$$

where $\phi$ is the Airy stress function that may be defined in the polar coordinates for convenience. Hence, the Laplace operator $\nabla^2$ becomes

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2}$$

The general solution of Eq. (1) was first given by Michell.[1]

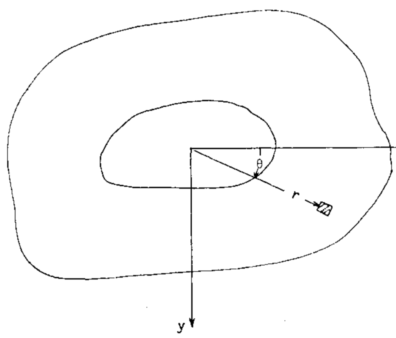


**Fig. 1 Arbitrarily shaped, doubly connected plate.**

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For an arbitrarily shaped doubly connected plate with the origin of the coordinate system defined inside of the hole (Fig. 1), the appropriate $\phi$ is reduced to

$$\phi = a_0 \log r + b_0 r^2 + a'_0\theta + (b_1 r^3 + a'_1 r^{-1})\cos\theta + (d_1 r^3 + c'_1 r^{-1})\sin\theta + \sum_{n=2}^{\infty}(a_n r^n + b_n r^{n+2} + c_n r^{-n} + d_n r^{-n+2})\cos n\theta + \sum_{n=2}^{\infty}(a'_n r^n + b'_n r^{n+2} + c'_n r^{-n} + d'_n r^{-n+2})\sin n\theta \tag{3}$$

Equation (3) is further reduced for plates with one or more lines of symmetry in both geometry and loadings. For simplicity, all plates that are treated in this Note as numerical examples have at least two lines of symmetry. Hence,

$$\phi = a_0 \log r + b_0 r^2 + \sum_{n=2,4\ldots}^{\infty}(a_n r^n + b_n r^{n+2} + c_n r^{-n} + d_n r^{-n+2})\cos n\theta \tag{4}$$

where $a_0$, $b_0$, $a_2$, . . . are determined from the prescribed boundary conditions. The boundary conditions prescribed are either normal or tangential stress or both. No restrictions are posted on how they may be mixed. With such a general case, the attempt of satisfying the boundary conditions exactly is out of the question.

The method employed here appeared to be used first by Morley[2] and later on extensively developed by Chen and Pickett.[3,4] The unknown coefficients are determined in such a way that the sum of the square of errors along both boundaries (for the number of terms chosen) is kept at a minimum.

Mathematically, let

$$U = \oint_{L_1+L_2}(\sigma_n - p)^2 ds + \oint_{L_1+L_2}(\tau_{nt} - q)^2 ds \tag{5}$$

where $p$ and $q$ are the prescribed boundary normal and tangential (shear) stresses and $\sigma_n$ and $\tau_{nt}$ are the correspond-

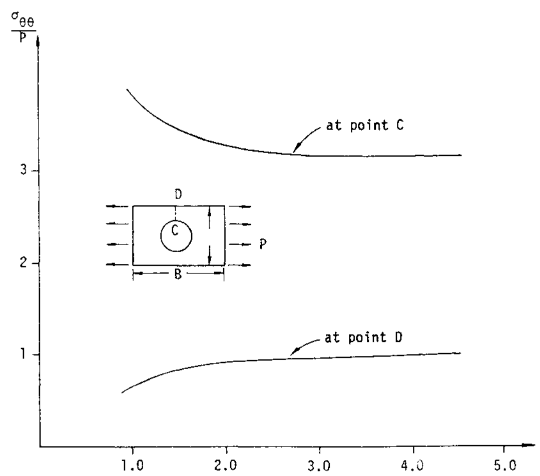


**Fig. 2 Stresses along the edges of rectangular plates.**

ing ones computed from the stress function $\phi$. When applying the principles of calculus of variation, we see that

$$(\delta U)_{\min} = 0 \tag{6}$$

Consequently, from Eq. (5)

$$\frac{\partial U}{\partial a_n} = 0, \quad \frac{\partial U}{\partial b_n} = 0, \quad \frac{\partial U}{\partial c_n} = 0, \quad \frac{\partial U}{\partial d_n} = 0 \tag{7}$$

$$n = 0, 2, 3, \ldots$$

A system of linear algebraic equation is then resulted from Eq. (7), from which constants $a_0$, $b_0$, $a_2$, . . . are solved accordingly.

The normal and tangential stress components $\sigma_n$ and $\tau_{nt}$, respectively, in terms of the stresses $\sigma_r$, $\sigma_\theta$, and $\tau_{r\theta}$ can be shown to be

$$\sigma_n = \frac{\sigma_r + \sigma_\theta}{2} + \frac{\sigma_r - \sigma_\theta}{2}\cos 2(\alpha - \theta) + \tau_{r\theta}\sin 2(\alpha - \theta)$$

$$\tau_{nt} = -\frac{\sigma_r - \sigma_\theta}{2}\sin 2(\alpha - \theta) + \tau_{r\theta}\cos 2(\alpha - \theta) \tag{8}$$

where $\sigma_r$, $\sigma_\theta$, and $\tau_{r\theta}$ are components of stresses in $r$ and $\theta$ directions,

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \quad \sigma_\theta = \frac{\partial^2\phi}{\partial r^2}, \quad \tau = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \tag{9}$$

In terms of $\phi$, Eqs. (8) become

$$\sigma_n = \frac{1}{2}\nabla^2\phi + \frac{1}{2}\left[\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial^2\phi}{\partial r^2}\right] \times \cos 2(\alpha - \theta) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\sin 2(\alpha - \theta)$$

$$\tau_{nt} = -\frac{1}{2}\left[\frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial^2\phi}{\partial r^2}\right] \times \sin 2(\alpha - \theta) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\cos 2(\alpha - \theta) \tag{10}$$

The sum of errors along both the outer ($L_1$) and inner ($L_2$) boundaries in Eq. (5) is usually done by numerical integration. A computer program was developed for the choice of $\phi$ given in Eq. (4).

### Results and Conclusion

Numerous cases were investigated and favorable results were obtained, but due to space limitations, only a few will be reported here. The problem of a thick-walled cylinder was solved by this method, and the results agreed within eight significant digits of the analytical solution. The results of stress concentrations of a perforated square plate were found to be within 3% of those given by Bailey and Hicks.[5] Figure 2 shows the results of a perforated rectangular plate. When the ratio of length to width is greater than three, the results approach rapidly those of Howland's[6] for an infinite perforated strip. The case of an elliptical plate having a central hole subjected to a uniform pressure at the inner boundary was also studied. Figure 3 shows the case of a circular hole whereas Fig. 4 shows an elliptical hole. Results in Fig. 3 compared favorably with those of Buchwald and Davis.[7] The stress distributions of hexagonal plates with various sizes of circular holes under a uniform tension are given in Fig. 5. These results are believed to be new.

The convergence to a solution by the least-square method is, in general, rapid. The accuracy of the solution can always be checked by examining the boundary conditions to determine whether or not they are being met to a satisfactory degree. The accuracy always improves when more terms

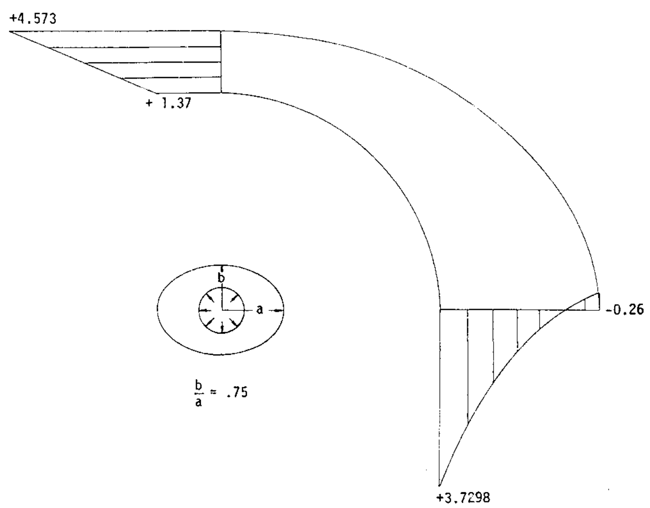


**Fig. 3 Stresses in an elliptical plate containing a circular hole.**

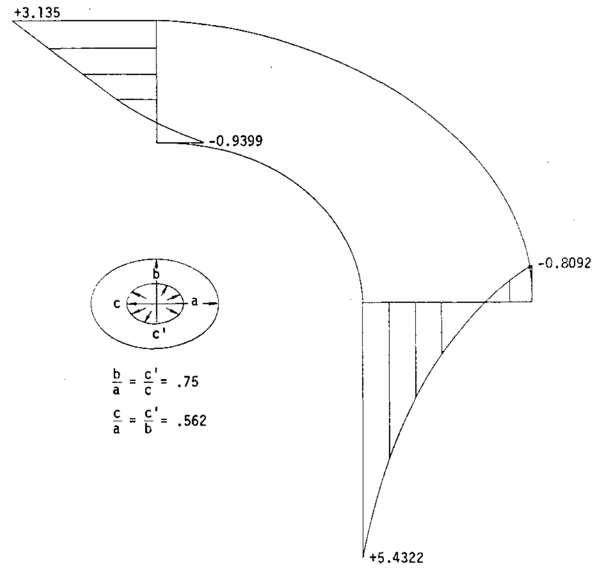


**Fig. 4 Stresses in an elliptical plate containing an elliptical hole.**

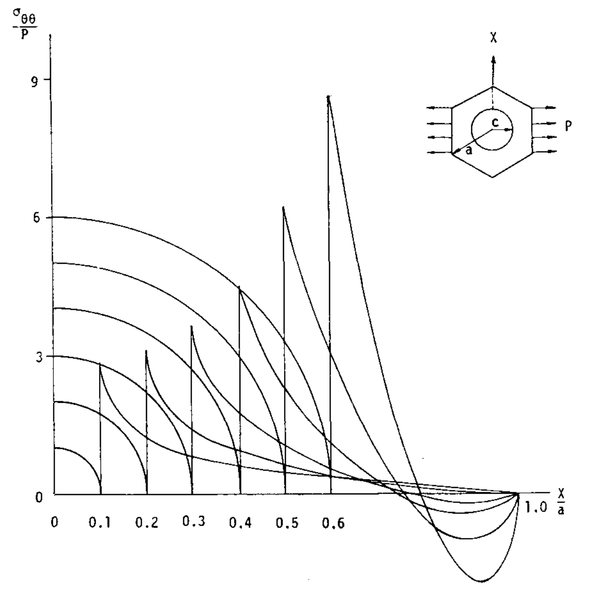


**Fig. 5 Stress distributions in a doubly connected hexagonal plate.**

of the series solution are taken, finer interval along the boundary perimeter is divided into, and more floating point digits are carried on the computer. One of the biggest advantages in using this method is that very little work is involved in switching from one shape of plate to the next or in changing from one type of boundary conditions to the other, however mixed. The former involves changing the portion of the program that computes the locations of boundary points, whereas the latter involves only the changing of an input data card.

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# Lift Reduction in Additive Solutions

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### Introduction

IT has been some time since the "rediscovery" (Hoyt and Fabula[1]) of Toms' phenomenon[2]: a large reduction of turbulent friction is obtained by adding extremely small quantities of high-molecular-weight additive to water. The potential practical application and the theoretical challenge have made this phenomenon one of the most important subjects for hydrodynamic research in recent years. Although an understanding of the physics governing this phenomenon is pursued actively at present,[3] the fact of large obtainable drag reduction is motivating engineers to look beyond the pipe-flow study and to undertake ship-model tests as well as prototype trials.

The motivation for using additives to reduce ship resistance is saving power or fuel from the economic point of view, and increasing speed from the performance point of view. However, the additive ejected around the ship will inevitably be introduced into the propeller section. The drag-reduction advantage can be greatly hampered if the additive produces adverse effects on the function of the propeller. This Note presents some observations, by-products of a drag reduction study,[4] on the possible thrust reduction of an impeller in additive solutions.

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### Experimental Technique

The experiment was conducted in a circulating water channel with a closed test section 44 in. long, 15 in. wide, and $7\frac{1}{2}$ in. deep; see Fig. 1a. At the upstream end of the contraction is a honeycomb. Energy dissipators consisting of screens and grids are placed at the discharge end of the test section. The water is circulated in the channel by a 3-bladed, 14-in.-diam impeller, placed in a circular duct and driven by a 3-hp motor-generator set.

A part of the cover plate at the test section, 10 in. wide and 20 in. long, is cut from the rest with a clearance of $\frac{1}{100}$ in. along four sides. The drag of this part of the cover plate is measured with a strain gage. The velocity in the channel is determined by photographing a small, neutrally buoyant particle, released at the upstream end of the test section. The time exposure picture of the path of the particle, illuminated by a strobe light, is shown as a series of bright dots on film. By comparing the distance between dots and the time interval between flashes, the channel speed is determined.

The test fluid consists of aqueous solutions of polyethylene oxide (Polyox WSR-301) of various concentrations. Shear degradation of the testing fluid is inevitable and will be discussed in a later section.

### Results

For the pure water case, the channel velocity at the test section, proportional to the velocity in the slipstream, is shown in Fig. 1b to vary linearly with the rotational speed of the impeller. The constant of proportionality can readily be determined. Part of the results of the drag-reduction study,[4] conducted simultaneously with the present experiment, are presented in Fig. 1c. These results provide some information about the present experimental conditions.

For tests with additive solutions, the ratio between the channel velocity and the rpm of the impeller was determined at various impeller speeds and compared with that of the pure water case (the proportionality constant determined previously). Several series of tests were conducted for each additive concentration. During each series, the channel speed

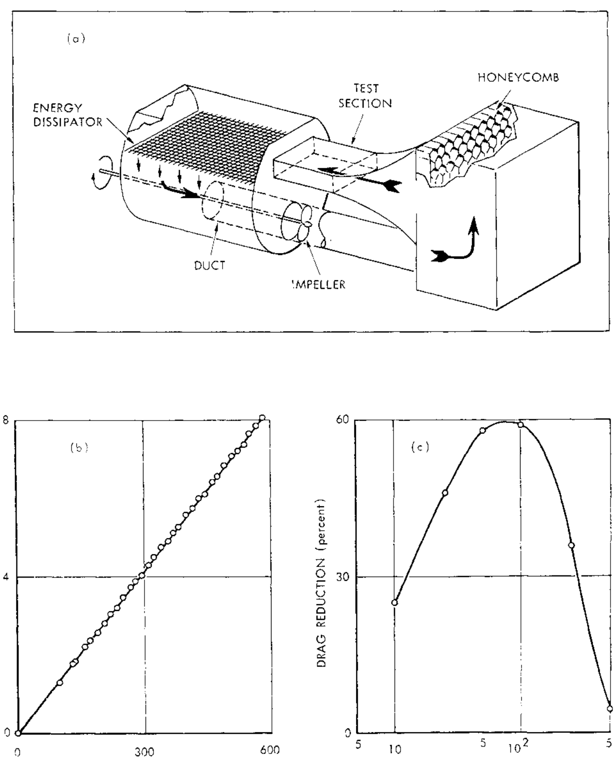


**Fig. 1 General view and calibration of equipment.**